EECE 574 - Adaptive Control

Stochastic Self-Tuning Controllers

Guy Dumont

Department of Electrical and Computer Engineering
University of British Columbia

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Stochastic Self-Tuning Controllers

- Minimum-variance control
- LQG control
- Predictive control
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Minimum-Variance Control

Consider the ARMAX model:

\[ A(q)y(t) = B(q)u(t) + C(q)e(t) \]

with

\[ A(q) = q^n + a_1q^{n-1} + \ldots a_n \]

\[ B(q) = b_0q^{n-d} + b_1q^{n-1-d} + \ldots b_{n-d} \]

\[ C(q) = q^n + c_1q^{n-1} + \ldots c_n \]

Note that \( d = \text{deg}\ A - \text{deg}\ B \) is the time delay of the process.

As usual, \( e(t) \) is assumed to be zero-mean white noise.
Omitting the argument $q$ in the various polynomials, the output at time $t + d$ can be written as:

$$y(t + d) = \frac{B}{A} u(t + d) + \frac{C}{A} e(t + d)$$

The second term in the right-hand side of the above equation consists of noise terms which at time $t$ are future and unknown, and of some past and present terms.

Because the noise $e(t)$ is white, those future noise terms cannot be predicted, i.e. $E[e(t + j)] = 0$ for $j > 0$. 
Minimum-Variance Control

We can express the noise filter $C/A$ by its impulse response model, and separate the unknown future terms from the known past and present ones:

$$
\frac{C}{A} e(t+d) = e(t+d) + f_1 e(t+d-1) + \ldots + f_{d-1} e(t+1) \\
\text{future unknown terms} \\
+ f_d e(t) + f_{d+1} e(t-1) + \ldots \\
\text{present and past terms}
$$
This can be rewritten as

\[ \frac{C}{A} q^{d-1} e(t+1) = Fe(t+1) + \frac{qG}{A} e(t) \]

i.e. \( F \) and \( G \) satisfy the following Diophantine equation with \( \deg F = d - 1 \) and \( \deg G = n - 1 \):

\[
q^{d-1} C(q) = A(q)F(q) + G(q)
\]

With that the predictor becomes

\[
y(t+d) = \frac{B}{A} u(t+d) + (Fq^{-d+1} + \frac{G}{A}q^{-d+1})e(t+d)
\]
Reconstructing the noise sequence $e(t)$ as $e(t) = [Ay(t) - Bu]/C$ we can write the $d$-steps ahead predictor

$$y(t + d) = \frac{B}{A} u(t + d) + \frac{qG}{A} e(t) + Fe(t + 1)$$

as

$$y(t + d) = \frac{B}{A} u(t + d) + \frac{qG}{A} \frac{Ay(t) - Bu(t)}{C} + Fe(t + 1)$$

or

$$y(t + d) = \frac{qG}{C} y(t) + \frac{B(Cq^d - qG)}{AC} u(t) + Fe(t + 1)$$

with $q^{d-1}C = AF + G$ this becomes

$$y(t + d) = \frac{qG}{C} y(t) + \frac{qBF}{C} u(t) + Fe(t + 1)$$
The predictor can then be written as

\[ y(t + d) = \hat{y}(t + d|t) + Fe(t + 1) \]

If \( y_r \) denotes the setpoint, then the minimum variance controller is obtained by setting

\[ \hat{y}(t + d|t) = y_r = \frac{qG}{C} y(t) + \frac{qBF}{C} u(t) \]

thus giving the controller:

**MVC Controller**

\[
    u(t) = \frac{C(q)}{B(q)F(q)} y_r - \frac{G(q)}{B(q)F(q)} y(t)
\]
Minimum-Variance Control

Note that when minimum-variance control is used, then

\[ y(t + d) = F e(t + 1) \]
\[ = e(t + d) + f_1 e(t + d - 1) + \ldots \]
\[ + f_{d-1} e(t + 1) \]

i.e.

- The output is a moving-average process of order \( d - 1 \).
- A characteristic of such a process is that its autocorrelation function vanishes after \( d \) lags.
- This is an important feature of minimum-variance control, which will prove very useful when testing the optimality of a stochastic control scheme.
Using this controller gives for the closed-loop system:

\[ BCq^{d-1}y(t) = BCy_r + BCFe(t) \]

Multiplying both sides of the Diophantine equation by \( B(q) \), we can interpret the minimum-variance controller as a pole-placement controller.

\[ q^{d-1}C(q)B(q) = A(q)F(q)B(q) + G(q)B(q) \]

The MVC can then be interpreted as a pole placement controller with \( B^+ = B, A_m = q^{d-1} \) and \( A_0 = C \)
Like the Dahlin controller, the minimum-variance controller cancels all process zeros, thus no zeros are allowed outside the unit circle, and poorly damped zeros will cause ringing.

If the $B$-polynomial is factored as

\[ B(q) = B^+(q)B^-(q) \]

with $B$ monic and where all zeros of $B^+(q)$ are within the unit circle, and those of $B^-(q)$ are on or outside the unit circle.
The minimum-variance control law is then given by:

\[ u(t) = -\frac{G(q)}{B^+(q)F(q)}y(t) \]

with \( \deg F(q) = d + \deg B^--1 \), \( \deg G(q) = n - 1 \) and

\[ q^{d-1}C(q)B^{-*}(q) = A(q)F(q) + B^-(q)G(q) \]

where \( B^{-*}(q) = q^{\deg B^-}B^-(q^{-1}) \) projects the unstable zeros inside the unit circle.
Minimum-Variance Control

- Major drawbacks of minimum-variance control
  - Hyperactivity
  - Lack of tuning parameter
- Modifications
  - Moving-average controller
  - Input weighting

\[ J = E[y^2 + \rho u^2] \]
\[ J = E[y^2 + \rho \Delta u^2] \]
The moving-average controller can be seen as a simplification of the above scheme, where instead of projecting the unstable zeros inside the unit circle by using the reciprocal of $B$ before cancelling them, we simply add an equivalent number of poles at the origin. The Diophantine equation is then

$$q^{h-1} C(q) = A(q)F(q) + B^-(q)G(q)$$

or

$$q^{h-1} B^+(q)C(q) = A(q)B^+(q)F(q) + B(q)G(q)$$

where $h = \deg A - \deg B^+$ and $\deg F = h - 1$. 
The control law is

\[ u(t) = -\frac{G}{B+F}y(t) \]

The closed-loop system is

\[ y(t) = \frac{CB^+F}{q^{h-1}B+C}e(t) \]
\[ = q^{-h+1}Fe(t) \]
\[ = (1+f_1q^{-1}+\ldots+f_{h-1}q^{-h+1})e(t) \]

The controlled output is then a moving-average of order \( h - 1 \)

- If \( h = d \) then minimum-variance control is obtained.
- If \( h \deg A = n \), then no zeros are cancelled.
Theorem (LQG Control)

Consider the process model

\[ A(q)y(t) = B(q)u(t) + C(q)e(t) \]

and the loss function

\[ J = \mathbb{E}\{(y(t) - y_r(t))^2 + \rho u^2(t)\} \]

Assume that there is no common factor to all three polynomials \( A(q), B(q) \) and \( C(q) \). The optimal control law that minimizes \( J \) is given by

\[ R(q)u(t) = -S(q)y(t) + T(q)y_r(t) \]

where \( T(q) = t_0 C(q) \) with \( t_0 = P(1)/B(1) \) and \( R \) and \( S \) satisfy the Diophantine equation

\[ A(q)R(q) + B(q)S(q) = P(q)C(q) \]

where the monic and stable \( P(q) \) is obtained from the spectral factorization

\[ rP(q)P(q^{-1}) = \rho A(q)A(q^{-1}) + B(q)B(q^{-1}) \]

For proof, see Aström and Wittenmark, Computer Controlled Systems
The LQG controller is obtained as solution of the Diophantine equation

\[ P(q)C(q) = A(q)R(q) + B(q)S(q) \]

The closed-loop characteristic polynomial is \( P(q)C(q) \) where the stable polynomial \( P(q) \) is obtained from the following spectral factorization:

\[ rP(q)P(q^{-1}) = \rho A(q)A(q^{-1}) + B(q)B(q^{-1}) \]
If $A$ and $B$ have a stable common factor $A_2$, i.e. $A = A_1A_2$ and $B = B_1A_2$ then $P$ can be written as $P = P_1A_2$ where $P_1$ satisfies

$$rP_1(q)P_1(q^{-1}) = \rho A_1(q)A_1(q^{-1}) + B_1(q)B_1(q^{-1})$$

and

$$P = P_1A_2$$

The Diophantine equation to be solved then becomes

$$P_1(q)C(q) = A_1(q)R(q) + B_1(q)S(q)$$

$A_2$ contains the **uncontrollable** modes
More complicated when some uncontrollable modes are unstable, i.e. when $A_2(q) = A_2^+(q)A_2^-(q)$

With a controller $u(t) = -S(q)/R(q)y(t)$ the closed-loop input and output are described by

$$y(t) = \frac{R(q)C(q)}{A(q)R(q) + B(q)S(q)}e(t)$$

$$u(t) = \frac{S(q)C(q)}{A(q)R(q) + B(q)S(q)}e(t)$$

The output $y$ will be bounded if $R(q)$ contains $A_2^-(q)$

The input $u$ will be bounded if $s(q)$ contains $A_2^-(q)$

Because $R$ and $S$ are co-prime, only $y$ will be bounded by making sure that $R(q)$ contains $A_2^-(q)$ as a factor. This is the internal model principle.
Indirect Stochastic STC

Indirect MV-STR

Straightforward design

- Use e.g. AML to obtain estimates $\hat{A}$, $\hat{B}$ and $\hat{C}$
- Compute zeros of $\hat{C}$ and project unstable ones inside the unit circle
- Factor $\hat{B}$ as $\hat{B} = \hat{B}^+ \hat{B}^-$
- Eliminate common factors between $\hat{A}$ and $\hat{B}^-$
- Solve the Diophantine equation

$$q^{d-1} \hat{C}(q) \hat{B}^{-\ast}(q) = \hat{A}(q)F(q) + \hat{B}^-(q)G(q)$$

where $\deg F(q) = d + \deg \hat{B}^- - 1$, $\deg G(q) = n - 1$ and $\hat{B}^{-\ast}(q) = q^{\deg \hat{B}^-} \hat{B}^-(q^{-1})$

- Apply the minimum variance controller

$$u(t) = -\frac{G(q)}{\hat{B}^+(q)F(q)}y(t)$$
Consider the system

\[ y(t+1) + ay(t) = bu(t) + e(t+1) + ce(t) \]

with \( a = -0.9, \ b = 3, \) and \( c = -0.3. \)

The minimum variance controller is

\[ u(t) = \frac{a - c}{b} y(t) = -s_0 y(t) = -0.2 y(t) \]

Initial estimates \( \hat{a}(0) = \hat{c}(0) = 0, \ \hat{b}(0) = 1 \)
Figure: Indirect MV-STR
Indirect Stochastic STC

Indirect LQG-STR

- Estimate $\hat{A}$, $\hat{B}$ and $\hat{C}$ using e.g. AML
- Compute zeros of $\hat{C}$ and project unstable ones inside the unit circle
- Eliminate common factors between $\hat{A}$, $\hat{B}$ and $\hat{C}$
- Find the common factor $A_2$ between $\hat{A}$ and $\hat{B}$
- Given $\rho$, solve the appropriate spectral factorization problem
- Solve the appropriate Diophantine equation
- Implement control law with computed $R$, $S$, and $T$

The solution of the spectral factorization problem is the major computation in the LQG STR. It is crucial to use a method that guarantees a stable $P$. 
Direct MV-STR Example

- Consider again the system

\[ y(t+1) + ay(t) = bu(t) + e(t+1) + ce(t) \]

with \( a = -0.9, b = 3, \) and \( c = -0.3. \)

- The minimum variance controller is

\[ u(t) = \frac{a - c}{b} y(t) = -s_0 y(t) = -0.2y(t) \]

- Consider a direct STR based on the model

\[ y(t+1) = r_0 u(t) + s_0 y(t) \]

for which the control law is

\[ u(t) = -\frac{\hat{s}_0}{\hat{r}_0} y(t) \]

with \( \hat{r}_0 = 1 \) and \( \hat{s}_0 \) is estimated with a simple RLS.
Direct MV-STR Example

Figure: Indirect MV-STR
Direct Stochastic Self-Tuning Controllers

- The previous indirect scheme involves many calculations at each step and requires the estimation of the $C$-polynomial.
- Direct schemes are much more economical from a computational viewpoint.
- Direct STR involves reparameterizing the problem in terms of the controller parameters.
- Key to this reparameterization is the predictor form.
The predictor for $y(t + d)$ derived for minimum-variance control can be written in terms of polynomials in the backward shift operator $q^{-1}$ as

$$y(t + d) = \frac{1}{C} (Gy(t) + BFu(t)) + Fe(t + d)$$

or

$$y(t + d) = S(q^{-1})y_f(t) + R(q^{-1})u_f(t) + F(q^{-1})e(t + d)$$

with

$$y_f(t) = \frac{y(t)}{C(q^{-1})}$$

In this predictor model, the controller polynomials $R$ and $S$ appear directly.
Direct MV-STR

- Given \( n \) and \( d \), then \( \text{deg} \, R = n + d - 1 \) and \( \text{deg} \, S = n - 1 \). Let \( \frac{1}{C^*(q^{-1})} \) be a stable filter.
- Get estimates \( \hat{R} \) and \( \hat{S} \) from

\[
y(t + d) = S(q^{-1})y_f(t) + R(q^{-1})u_f(t) + \varepsilon(t + d)
\]

where

\[
y_f(t) = \frac{y(t)}{C(q^{-1})}
\]

using RLS with

\[
\varphi^T(t) = \frac{1}{C}[u(t) \ldots u(t - n - d + 1) y(t) \ldots y(t - n + 1)]
\]

\[
\theta^T = [r_0 \ldots r_{n+d-1} s_0 \ldots s_{n-1}]
\]

- Apply the control law

\[
\hat{R}(q^{-1})u(t) = -\hat{S}(q^{-1})y(t)
\]
Properties of the Direct MV-STR

Assume direct MV-STR with $C^*(q^{-1}) = 1$

**Property 1:** If regression vectors are bounded, then the closed-loop system is such that

$$y(t + \tau)y(t) = 0 \quad \tau = d, \ldots d + n - 1$$

$$y(t + \tau)u(t) = 0 \quad \tau = d, \ldots d + n + d - 1$$

**Property 2:** If the direct MV-STR is applied to

$$A(q)y(t) = B(q)u(t) + C(q)e(t)$$

then

$$y(t + \tau)y(t) = 0 \quad \tau = d, \ldots$$

Thus, if the direct MV-STR converges, and if there are sufficiently many parameters in the controller, then it converges to the minimum-variance controller.
Some Remarks

The predictor can be written as

\[ y(t + d) = \varphi^T(t)\hat{\theta}(t) + \epsilon(t + d) \]

and the MVC controller as

\[ \varphi^T(t)\hat{\theta}(t) = 0 \]

**Remark 1:** Note that \( k\varphi^T(t)\hat{\theta}(t) = 0 \) also gives MV control. Parameters have one degree of freedom and may thus wander in unison (they lie on a linear manifold). Unique estimation may be obtained by fixing e.g. \( r_0 \). Ideally, it should be equal to \( b_0 \)

**Remark 2:** For convergence, it is necessary that \( \{u(t)\} \) be persistently exciting of order \( \geq 2n + d \). How can we guarantee that in adaptive control? How can we even guarantee stability? These are non-trivial questions.
Some Remarks

**Remark 3:** It is easy to add feedforward and command signals

\[ y(t + d) = S(q^{-1})y_f(t) + R(q^{-1})u_f(t) + S_{ff}(q^{-1})v_f(t) + \varepsilon(t + d) \]

where \( v_f \) is the filtered measured disturbance.

The controller is then

\[ \hat{R}(q^{-1})u(t) = -\hat{S}(q^{-1})y(t) - \hat{S}_{ff}(q^{-1})v(t) \]

Can also add setpoint tracking.

**Remark 4:** By replacing \( d \) by \( h > d \), we obtain a self-tuning moving-average controller. By sufficiently large \( h \), no zeros are cancelled, and non-minimum phase systems can be controlled.
Consider an integrator with time delay $\tau$ sampled with sampling period $h$.

$$A(q) = q(q - 1)$$

$$B(q) = (h - \tau)(q + \frac{\tau}{h - \tau})$$

$$C(q) = q(q + c)$$

- Minimum phase if $\tau < h/2$.
- With $h = 1$, at time $t = 100$, the delay is changed from 0.4 to 0.6.
- Simulation with $d = 1$ (MV-STR) and $d = 2$ (MA-STR).
Direct MA-STR Example

Figure: Direct MV-STR and MA-STR
Generalized MV Self-Tuners

- Extended Minimum-Variance Controller
- Generalized Minimum-Variance Controller

### Known Parameters

Define the auxiliary variable

\[ \phi(t) \triangleq Py(t) - Rq^{-d}y_r(t) + Qq^{-d}u(t) \]

where \( P, Q \) and \( R \) are polynomials.

Multiplying by \( A \) and assuming that \( y(t) \) is governed by an ARMAX process gives

\[
A\phi(t) = P[Bq^{-d}u(t) + Ce(t)] - q^{-d}ARy_r(t) + Qq^{-d}Au(t)
\]

or

\[
\phi(t) = \frac{q^{-d}[PB + QA]}{A}u(t) - q^{-d}Ry_r(t) + \frac{PC}{A}e(t)
\]
Consider the Diophantine equation

\[ PC = AF + q^{-d}G \]

where \( \deg F = d - 1 \) and \( \deg G = n + n_p - 1 \) With this, we can write a predictor for \( \phi(t+d) \):

\[ \phi(t+d) = \frac{BF + QC}{C} u(t) + \frac{G}{C} y(t) - Ry_r(t) + Fe(t+d) \]

or

\[ \phi(t+d) = \hat{\phi}(t+d|t) + \varepsilon(t+d) \]

with

\[ \hat{\phi}(t+d|t) = \frac{1}{C} [(BF + QC)u(t) + Gy(t) - CRy_r(t)] \]
Extended Minimum-Variance Control

The error term \( \epsilon(t + d) = Fe(t + d) \) is uncorrelated with \( \hat{\phi}(t + d | t) \). We can thus proceed as in minimum variance control. The cost function

\[
J = E[\phi^2(t + d)]
\]

is minimized by setting \( \hat{\phi}(t + d | t) = 0 \), i.e.

\[
(BF + QC)u(t) + Gy(t) - CRy_r(t) = 0
\]

or

\[
u(t) = -\frac{G}{BF + QC}y(t) + \frac{CR}{BF + QC}y_r(t)
\]

then

\[
\phi(t + d) = Fe(t + d)
\]

The above control law minimizes \( E[\phi^2(t + d)] \), i.e. is the MV controller for \( \phi \).
The closed-loop system is

\[ y(t) = \frac{B}{A} q^{-d} \left[ \frac{C R y_r(t) - G y(t)}{B F + Q C} \right] + \frac{C}{A} e(t) \]

or

\[ C [B P + Q A] y(t) = B C R q^{-d} y_r + c [B F + Q C] e(t) \]

The closed-loop characteristic polynomial is

\[ C [B P + Q A] \]

EMVC can handle non-minimum phase systems.
Extended Minimum-Variance Control

The EMVC control law that minimizes

\[ E[\phi^2(t + d)] = E[(Py(t + d) - Ry_r + Qu(t))^2] \]

also minimizes

\[ E[(Py(t + d) - Ry_r)^2 + \mu (Qu(t))^2] \quad \mu = \frac{b_0}{q_0} \]

i.e. this can be interpreted as LQG!

_Proof:_ Consider

\[ I = E[(Py(t + d) - Ry_r)^2 + \mu (Qu(t))^2] \]
\[ = ((P\hat{y}(t + d|t) - Ry_r)^2 + \mu (Qu(t))^2 + \sigma_e^2) \]

Then

\[ \frac{dI}{du(t)} = 2(P\hat{y}(t + d|t) - Ry_r)b_0 + 2\mu q_0 Qu(t) = 0 \]

Implies

\[ P\hat{y}(t + d|t) + \mu \frac{q_0}{b_0} Qu(t) - Ry_r = 0 \]

which is the same control as before if \( \mu = b_0/q_0 \).
Extended Minimum-Variance Control

Note that zero steady-state error requires

\[
\frac{BR}{PB + QA}\bigg|_{z=1} = 1
\]

This is satisfied by choosing \( R = P(1) \) and \( Q(1) = 0 \)

- With \( P = R = 1 \) and \( Q = \lambda \)
  \[
  J = E[(y(t+d) - y_r)^2 + \lambda' u^2(t)]
  \]

- With \( P = R = 1 \) and \( Q = \lambda (1 - q^{-1}) \)
  \[
  J = E[(y(t+d) - y_r)^2 + \lambda' \Delta u^2(t)]
  \]
  the controller then contains an integrator.

- With \( P = R = 1 \) and \( Q = 0 \)
  \[
  J = E[(y(t+d) - y_r)^2]
  \]

MVC with setpoint tracking.
A direct EMV self-tuner can be implemented by estimating directly the parameters of the predictor form:

$$C\hat{\phi}(t + d|t) = (BF + QC)u(t) + Gy(t) - CRy_r(t)$$

Note that this seemingly requires direct estimation of the $C$ polynomial. However, the predictor

$$C\phi(t + d) = (BF + QC)u(t) + Gy(t) - CRy_r(t) + FCe(t + d)$$

can be rewritten as

$$[C + (1 - C)]\phi(t + d) = (BF + QC)u(t) + Gy(t) - CRy_r(t) + FCe(t + d) + (1 - C)\phi(t + d)$$

$$\phi(t + d) = (BF + QC)u(t) + Gy(t) - CRy_r(t) + Fe(t + d) + (1 - C)[\phi(t + d) - Fe(t + d)]$$

Under the previously derived control law, $\phi(t + d) = Fe(t + d)$ and the last term of the above equation vanishes.

Note that it only says that convergence to the correct control law is possible even without knowing $C$, but it does not establish that convergence. This will be discussed later.
Example

\[ y(t) = 2y(t-1) + u(t-2) + 2u(t-3) + e(t) \text{ where } \sigma_e^2 = 0.5 \]

**Figure:** MVC
Example

\[ y(t) = 2y(t - 1) + u(t - 2) + 2u(t - 3) + e(t) \text{ where } \sigma_e^2 = 0.5 \]

Figure: EMVC, \( P = Q = 1, BP + AQ = 2 \)
Example

\[ y(t) = 2y(t - 1) + u(t - 2) + 2u(t - 3) + e(t) \text{ where } \sigma_e^2 = 0.5 \]

**Figure:** Adaptive EMVC, \( P = Q = 1, BP + AQ = 2 \)
EMVC Example

\[ y(t + 1) + ay(t) = bu(t) + e(t + 1) + ce(t) \] with \( a = -0.9 \), \( b = 3 \), and \( c = -0.3 \).

The output variance as a function of the input variance is as shown below.
EMVC Example

\[
\sum y^2(k)
\]

\[
\sum u^2(k)
\]