EECE 574 - Adaptive Control
Other Approaches to Adaptive Control

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Outline

- Novel way of expressing controller transfer function
- Provides new insight into control design
- The key feature of the new parameterization is that it renders the closed loop sensitivity functions \textbf{linear} or \textit{(more correctly, affine)} in a design variable
- We call it the \textbf{affine parameterization}
- The so-called Youla-Kucera parametrization now plays a central role in control, identification and adaptive control
Recall that control implicitly and explicitly depends on plant model inversion. This is best seen in the case of open loop control.

In open loop control the input, $U(s)$, is generated from the reference signal $R(s)$, by a transfer function $Q(s)$, i.e. $U(s) = Q(s)R(s)$.

This leads to an input-output transfer function of the following form:

$$T_0(s) = P(s)Q(s)$$
This simple formula highlights the fundamental importance of inversion, as $T_0(j\omega)$ will be 1 only at those frequencies where $Q(j\omega)$ inverts the model. Note that this is consistent with the prototype solution to the control problem described earlier.

A key point is that $T_0(s) = P(s)Q(s)$ is affine in $Q(s)$.

On the other hand, with a conventional feedback controller, $C(s)$, the closed loop transfer function has the form

$$T_0(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}$$

The above expression is nonlinear in $C(s)$. 
Comparing the two previous equations, we see that the former affine relationship holds if we simply parameterize $C(s)$ in the following fashion:

$$Q(s) = \frac{C(s)}{1 + C(s)P(s)}$$

We can then design in terms of $Q(s)$ and then obtain $C(s)$ from $Q(s)$ and $P(s)$.

This is the essence of the idea presented here.
Affine Parameterization. The Stable Case

- We can invert the relationship given on the previous slide to express $C(s)$ in terms of $Q(s)$ and $P(s)$:

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)}$$

- We will then work with $Q(s)$ as the design variable rather than the original $C(s)$.

- Note that the relationship between $C(s)$ and $Q(s)$ is one-to-one and thus there is no loss of generality in working with $Q(s)$. 
Actually a very hard question is the following:

Given a stable transfer function $P(s)$, describe all controllers, $C(s)$ that stabilize this nominal plant.

However, it turns out that, in the $Q(s)$ form, this question has a very simple answer, namely all that is required is that $Q(s)$ be stable.

This result is formalized in the lemma stated on the next slide.
Lemma (Lemma 15.1: Affine parameterization for stable systems)

Consider a plant having a stable nominal model $G_0(s)$ controlled in a one d.o.f. feedback architecture with a proper controller.

Then the nominal loop is internally stable if, and only if, $Q(s)$ is any stable proper transfer function when the controller transfer function $C(s)$ is parameterized as

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)}$$
Proof
We note that the four sensitivity functions can be written as

\[
\begin{align*}
T_0(s) &= Q(s)P(s) \\
S_0(s) &= 1 - Q(s)P(s) \\
S_{i0}(s) &= [1 - Q(s)P(s)]P(s) \\
S_{u0}(s) &= Q(s)
\end{align*}
\]

We are for the moment only considering the case when \(P(s)\) is stable. Then, we see that all of the above transfer functions are stable if, and only if, \(Q(s)\) is stable.
Youla’s parameterization of all stabilizing controllers for stable plants

This particular form of the controller, i.e.

\[ C(s) = \frac{Q(s)}{1 - Q(s)P(s)} \]

can be drawn schematically below.
Conversely let $P_0(s) = N(s)D^{-1}(s)$ be a nominal plant stabilized by the controller $C(s) = X(x)Y^{-1}(s)$.

Then all the plants stabilized by $C(s)$ are given by

$$P(s) = \frac{N(s) + Y(s)S(s)}{D(s) - X(s)S(s)}$$

where $S(s)$ is a stable proper rational transfer function.
Closed-Loop Identification
Fig. 12. Details of closed-loop identification.
SEEKER: How do I gain Good Judgement?
TEACHER: You must first acquire Wisdom.
SEEKER: How do I gain Wisdom?
TEACHER: From Bad Judgement.
**Step 1**

Set $G_i = G_0$, where $G_0$ is the transfer function of an initial model of the plant.

**Step 2**

Factorize $G_i$ as

$$G_i = [G_i]_m [G_i]_a$$

where $[G_i]_m$ is the minimum phase factor of $G_i$ with a relative degree of $n$ and $[G_i]_a$ is the associated allpass factor of $G_i$. 
Step 3

For $j = 1$ find

$$K_{j,i} = \frac{Q_{j,i}}{1 + Q_{j,i}G_i}$$

with

$$Q_{j,i} = \left[G_i\right]^{-1}F_{j,i}$$

where the parameter $\lambda_{j,i}$ in the transfer function

$$F_{j,i} = \left(\frac{\lambda_{j,i}}{s + \lambda_{j,i}}\right)^{\eta+1}$$

is chosen such that $K_{j,i}$ robustly stabilizes $G_i$ in the sense that the filtered (noisy) step response of the actual closed-loop system has, at most, few oscillations and/or overshoots. Stop here if such a robust stabilizing controller cannot be found. Also stop here if the robust stabilizing controller results in a closed-loop system which meets the specified bandwidth. Otherwise proceed to the next step.
Step 4

Let \( j = j + 1 \) and set \( \lambda_{j,i} = \lambda_{j-1,i} + \epsilon \) for small \( \epsilon > 0 \) and redesign the controller \( K_{j,i} \) using the equations given in Step 3. Stop here if the design produces a robust stabilizing controller with the closed-loop system satisfying the specified bandwidth. Otherwise repeat this step if \( K_{j,i} \) robustly stabilizes \( G_i \); else proceed to the next step.
Step 5

Perform control-relevant system identification to obtain $\hat{r}_{j,i}$. For this purpose we apply an algorithm such as least squares to obtain an estimate $\hat{r}_{j,i}$ of $R_{j,i}$ which satisfies

$$\beta_1 = R_{j,i} \alpha_2 + v$$

This depends on using the signals

$$\beta_1 = Y_{j,i}(y - G_i u), \quad \alpha_2 = R_{j,i} Y_{j,i} X_{j,i} r_1$$

(We actually used discrete time samples of $\beta_1$ and $\alpha_2$ and an output error algorithm to construct a strictly causal second-order estimate from which a continuous time strictly proper $\hat{r}_{j,i}$ was obtained.) Using $\hat{r}_{j,i}$, the model is updated via the following set of equations:

$$\bar{R}_{j,i} = [G_i]_m (s + \lambda_{j,i})^n, \quad r_{j,i} = \bar{R}_{j,i} \hat{r}_{j,i}, \quad G_{i+1} = G_i + \frac{r_{j,i}}{1 - r_{j,i} Q_{j,i}}$$
Step 6

If $G_{i+1}$ is stable, find the reduced-order model

$$
\hat{G}_{i+1} = \arg \min \left\{ \frac{G_{i+1}K_{j,i}}{1 + G_{i+1}K_{j,i}} - \frac{\eta K_{j,i}}{1 + \eta K_{j,i}} \right\}_\infty
$$

Otherwise stop here.

Step 7

Set $G_i = \hat{G}_{i+1}$ and return to Step 2.

Remarks

(i) In the algorithm, system identification has to be carried out when

$$
\| T_{N,i} - \bar{T}_{N,i} \|_\infty
$$

is no longer small. Broadly speaking, this will correspond to a significant difference between the designed nominal performance (depending on $G_i$ and $K_{N,i}$) and the actual performance
With reference to Figure 3, we shall present some simulation results of applying the windsurfer approach to the control of a system with

\[ G(s) = \frac{9}{(s + 1)(s^2 + 0.06s + 9)}, \quad H(s) = 1 \]

and \( \epsilon \) as zero-mean disturbance with a constant energy density of 0.0025 from 0 to 100 Hz.
The simulation results are presented in Figures 6–8. We start with an initial model which has the transfer function

\[ G_0 = \frac{0.8}{s + 1.2} \]

In all these figures the graphs on the left show the noisy unit-step responses of the actual closed-loop systems and those on the right show the corresponding lowpass-filtered signals. Graphs (a) and (b) of Figure 6 show the responses of the actual closed-loop system with a nominal bandwidth of 0.1 rad s\(^{-1}\). Note that overshoots and oscillations are absent for the response in graph (b). Graphs (c) and (d) of Figure 6 are for a nominal closed-loop bandwidth of 0.5 rad s\(^{-1}\). Note that the response in graph (d) is oscillatory and any attempt to increase the nominal closed-loop bandwidth further is likely to lead to instability. At this stage it is necessary to improve the accuracy of the model if we wish to increase the nominal closed-loop bandwidth further. To ensure that the signals are sufficiently exciting, low-amplitude sinusoids in the relevant frequency range are superimposed on the unit-step input just prior to system identification. The responses are shown in graphs (a) and (b) of Figure 7. The updated model
identification. The responses are shown in graphs (a) and (b) of Figure 7. The updated model has a transfer function

\[ G_1 = \frac{0.062528s^2 - 0.33968s + 10.279}{s^3 + 1.2801s^2 + 9.1173s + 10.324} \]

The updated model \( G_1 \) is used to redesign a nominal closed-loop system with a bandwidth of 0.51 rad s\(^{-1}\) and the responses are shown in graphs (c) and (d) of Figure 7. By comparing graph (d) of Figure 7 with that of Figure 6, we observe that the response no longer has oscillations. We also notice that the rise time in graph (d) of Figure 7 is about twice that in
The Windsurfer’s Approach

Graph (a)

Graph (b)

Graph (c)

Graph (d)
Classical MMAC - Athans, 1977

MultiModel Adaptive Control

\[
\begin{align*}
\xi(t) & \quad \theta(t) \\
u(t) & \quad y(t) \\
\end{align*}
\]

Unknown plant

KFs

KF #1

KF #2

KF #N

LQ-gains

\[-G_1\]

\[-G_2\]

\[-G_N\]

Posterior Probability Evaluator (PPE)

Residual covariances

\[S_1\]

\[S_2\]

\[\vdots\]

\[S_N\]

Posterior hypotheses probabilities

\[P_1(t)\]

\[P_2(t)\]

\[\vdots\]

\[P_N(t)\]
Switching MMAC - Hespanha, 2001

Multi-Model Adaptive Control

Disturbances \( \xi(t) \) \quad Sensor errors \( \theta(t) \)

\( u(t) \) : Controls \quad y(t) : Measurements

Unknown plant

Multi-estimators \( E_1(s) \) \( e_1(t) \) Prediction errors

\( E_2(s) \) \( e_2(t) \) : 

\( \cdots \)

\( E_N(s) \) \( e_N(t) \) : 

Monitoring signal generator

Multi-controllers \( C_1(s) \) \( u_1(t) \)

\( C_2(s) \) \( u_2(t) \) : 

\( \cdots \)

\( C_N(s) \) \( u_N(t) \)

Switching logic and dwell time

Switching indicator

\( \mu_1(t) \) \( \mu_2(t) \) : 

\( \cdots \)

\( \mu_N(t) \)
Material from:

- Lecture 1 from A Short Course on L1-Adaptive Control, at Lund University, N. Hovakimyan, http://www.control.lth.se/previouscourse
Adaptive controllers can behave well in idealized cases
But not be robust
  - Parameter drift in the presence of unmodelled dynamics and noise
Issues due to persistence of excitation
  - Instability or bursting
High gain control leading to small robustness margins
The control objective might be inappropriate

“If a plant is initially unknown or only partially unknown, a designer may not know a priori that a proposed design objective is or is not practically obtainable for the plant.”

---

*B. Anderson, Failures of Adaptive Control Theory, Communication in information and systems, 2005*

- The controller that minimizes the optimization objective might destabilize the system.
Main features of $L_1$ adaptive control

- Separation (decoupling) between adaptation and robustness
- Guaranteed fast adaptation
- Guaranteed transient response for system’s input and output
  - NOT achieved via persistence of excitation or gain scheduling
- Guaranteed (bounded away from zero) time-delay margin
- Uniform scaled transient response dependent on changes in initial conditions, unknown parameters, and reference input
- Suitable for development of theoretically justified Verification and Validation tools for feedback systems
The 1-norm of a vector $u = [u_1, \ldots, u_m]^T \in \mathbb{R}^m$ is defined as

$$\|u\|_1 \triangleq \sum_{i=1}^m |u_i|$$

The induced 1-norm

$$\|A\|_p \triangleq \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

is defined as

$$\|A\|_1 \triangleq \max_{1 \leq j \leq m} \sum_{i=1}^n |a_{ij}| \text{ (column sum)}$$
The space of piecewise-continuous integrable functions with bounded $L_1$-norm

$$\|f\|_{L_1} \triangleq \int_{0}^{\infty} \|f(\tau)\| d\tau < \infty$$

is denoted $L_1^n$, where any of the vector norms can be used.

**Lemma**

A continuous-time LTI (proper) system $y(s) = G(s)u(s)$ with impulse response matrix $g(t)$ is BIBO stable if and only if its $L_1$-norm is bounded, i.e. $\|g\|_{L_1} < \infty$ or equivalently $g(t) \in L_1$. 
Model Reference Control

- $K(s, \theta)$ designed such that closed-loop resembles $M_r(s)$.

Model Reference Adaptive Control

- Adapt $K(s, \hat{\theta})$ to match $M_r(s)$. 
\( K(s, \theta) \) is a state-feedback controller.

Adaptive controller parameter: \( \hat{\theta} \).
MRAC state feedback

Structural differences:
- State predictor $M_r$.
- Low-pass filter $C(s)$.

$L_1$ state feedback adaptive control
MRAC architecture with state observer

\[ e(t) = \hat{x}(t) - x(t) \]
\[ = k_g M_r r(t) + K(\hat{\theta}) x(t) - K(\hat{\theta}) x(t) - x(t) \]
\[ = k_g M_r r(t) - x(t) \]

- Structure is equivalent to MRAC state feedback.
- Enables insertion of low-pass filter \( C(s) \).
Insertion of low-pass filter $C(s)$

- Compensation only for uncertainties within the bandwidth of $C(s)$.
- “The L1 adaptive controller pursues a less ambitious, yet practically achievable, objective, namely, compensation of only the low-frequency content of the uncertainty within the bandwidth of the control channel.”

---

1N. Hovakimyan et al., IEEE Control Systems Magazine, Oct 2011
Consider the class of systems

\[ \dot{x}(t) = Ax(t) + b(u(t) + \theta^T x(t)), \quad x(0) = x_0, \]
\[ y(t) = c^T x(t) \]

- \( \theta \) is unknown, belongs to a given compact convex set \( \theta \in \Theta \).
- \( u(t) = u_m(t) + u_{ad}(t), \quad u_m(t) = -k_m^T x(t) \)
  - \( k_m \) renders \( A_m \triangleq A - bk_m^T \) Hurwitz
  - \( u_{ad}(t) \) is the adaptive component
- The predictor \( M_r \) is given by

\[ \hat{x}(t) = A_m \hat{x}(t) + b(u_{ad}(t) + \hat{\theta}^T x(t)), \quad x(0) = x_0, \]
\[ y(t) = c^T \hat{x}(t) \]
Can no longer achieve $M_r$
Compensate only for uncertainties within the bandwidth of $C(s)$
Non-adaptive closed-loop reference system
Projection-type adaptive law:

\[ \dot{\theta}(t) = \Gamma \text{Proj}(\hat{\theta}(t), -\hat{x}^T(t)Pbx(t)) \]

- \( \Gamma \) is the adaptation gain
- \( P = P^T > 0 \) solves the algebraic Lyapunov equation \( A_m^T P + PA_m = -Q \)
- The projection is confined to the set \( \Theta \)
$k_m$ and $C(s)$ verify the following $\mathcal{L}_1$-norm condition:

$$
\lambda \triangleq \| (sI - A_m)^{-1} b (1 - C(s)) \|_{\mathcal{L}_1} L = \| G(s) \|_{\mathcal{L}_1} L < 1,
$$

$$
L \triangleq \max_{\theta \in \Theta} \| \theta \|_1
$$

**Lemma**

If $\| G(s) \|_{\mathcal{L}_1} L < 1$, the closed-loop reference system is bounded-input bounded-state (BIBS) stable with respect to $r(t)$ and $x_0$.

Proof through small-gain theorem.
Lemma

The prediction error \( \tilde{x}(t) \) is uniformly bounded:

\[
\|\tilde{x}\|_{L_\infty} \leq \sqrt{\frac{\theta_{\text{max}}}{\lambda_{\text{min}}(P)\Gamma}}, \quad \theta_{\text{max}} \triangleq 4\max_{\theta \in \Theta} \|\theta\|^2,
\]

where \( \lambda_{\text{min}}(P) \) is the minimum eigenvalue of \( P \).

Lemma

\[
\lim_{t \to \infty} \tilde{x}(t) = 0
\]
$L_1$ conditions for stability and performance

**Theorem**

\[
\|x_{\text{ref}} - x\|_{L_\infty} \leq \frac{\gamma_1}{\sqrt{\Gamma}}, \quad \|u_{\text{ref}} - u\|_{L_\infty} \leq \frac{\gamma_2}{\sqrt{\Gamma}},
\]

\[
\lim_{t \to \infty} \|x_{\text{ref}}(t) - x(t)\| = 0, \quad \lim_{t \to \infty} \|u_{\text{ref}}(t) - u(t)\| = 0,
\]

where

\[
\gamma_1 \triangleq \frac{\|C(s)\|_{L_1}}{1 - \|G(s)\|_{L_1} L \sqrt{\frac{\theta_{\text{max}}}{\lambda_{\text{min}}(P)}}},
\]

\[
\gamma_2 \triangleq \|H_1(s)\|_{L_1} \sqrt{\frac{\theta_{\text{max}}}{\lambda_{\text{min}}(P)}} + \|C(s)\theta^T + k_m\|_{L_1} \gamma_1.
\]
The *closed-loop reference system* depends on the unknown $\theta$.

The *design system* is independent of $\theta$. 

\[
\begin{array}{c}
r \\
\downarrow \\
\circ \\
\downarrow \\
k_g \\
\downarrow \\
\circ \\
\downarrow \\
C(s) \\
\downarrow \\
\circ \\
\downarrow \\
u_{\text{des}} \\
\downarrow \\
\circ \\
\downarrow \\
(sI - A)^{-1}b \\
\downarrow \\
\circ \\
\downarrow \\
\theta^T \\
\downarrow \\
\circ \\
\downarrow \\
k_m^T \\
\downarrow \\
\circ \\
\downarrow \\
\theta^T \\
\end{array}
\]
\textbf{ Achieving desired specifications: }

- System output:
  \[ \| y_{\text{ref}} - y_{\text{des}} \|_{\infty} \leq \frac{\lambda}{1 - \lambda} \left\| c^T \right\|_{L_1} \left\| k_g (sI - A_m)^{-1} bC(s) \right\|_{L_1} \| r \|_{L_\infty} \]

- System input:
  \[ \| u_{\text{ref}} - u_{\text{des}} \|_{\infty} \leq \frac{\lambda}{1 - \lambda} \left\| C(s)\theta^T \right\|_{L_1} \left\| k_g (sI - A_m)^{-1} bC(s) \right\|_{L_1} \| r \|_{L_\infty} \]

\textbf{ Sufficient condition for stability: }

\[ \lambda = \left\| (1 - C(s))(sI - A_m)^{-1} b \right\|_{L_1} L < 1 \]

\textbf{ Performance improvement: }

\[ \lambda \rightarrow \min \]
$\| (1 - C(s)) (sI - A_m)^{-1} b \|_{\mathcal{L}_1}$ can be rendered arbitrarily small.

- Increase bandwidth of $C(s)$. “However, ... high bandwidths may result in high-gain feedback and thus lead to closed-loop systems with overly small robustness margins and susceptible to measurement noise.”
- Reduce bandwidth of $M_r$, or $(sI - A_m)^{-1} b$. “a certain amount of performance is sacrificed to maintain a desired level of robustness”
\( \mathcal{L}_1 \) controller design

- Use large adaptive gain (\( \Gamma \) aling):
  \[
  \| y - y_{\text{ref}} \|_{\mathcal{L}_\infty} \leq O \left( \frac{1}{\sqrt{\Gamma}} \right) \quad \| u - u_{\text{ref}} \|_{\mathcal{L}_\infty} \leq O \left( \frac{1}{\sqrt{\Gamma}} \right)
  \]

- Design \( C(s) \) to render \( \lambda \) sufficiently small (trade-off robustness for performance):
  \[
  \| y_{\text{ref}} - y_{\text{des}} \|_{\mathcal{L}_\infty} \leq O(\lambda) \quad \| u_{\text{ref}} - u_{\text{des}} \|_{\mathcal{L}_\infty} \leq O(\lambda)
  \]

Decoupling of Adaptation from Robustness

\[
\| y - y_{\text{des}} \|_{\mathcal{L}_\infty} \leq O \left( \frac{1}{\sqrt{\Gamma}} \right) + O(\lambda) \quad \| u - u_{\text{des}} \|_{\mathcal{L}_\infty} \leq O \left( \frac{1}{\sqrt{\Gamma}} \right) + O(\lambda)
\]
To summarize:

- The *closed-loop reference system* is stable if the conditions are satisfied.
- The $\mathcal{L}_\infty$-norm of the state error and the input error between the *closed-loop reference system* and the *adaptive system* is bounded.
- The $\mathcal{L}_\infty$-norm of the state error and the input error between the *closed-loop reference system* and the *design system* is bounded.
  - The transient response is therefore bounded.
- The error can be made arbitrarily small by increasing the gain $\Gamma$ and designing $C(s)$ and $M_r$. 
L_1 controller design: summary

- **State-Feedback:**
  - L_1 Adaptive Control for Systems with TV Parametric Uncertainty and TV Disturbances
  - L_1 Adaptive Control for Systems with Unknown System Input Gain
  - L_1 Adaptive Control for a class of Systems with Unknown Nonlinearities
  - L_1 Adaptive Control for Nonlinear Systems in the presence of Unmodeled Dynamics
  - L_1 Adaptive Control for Systems in the presence of Unmodeled Actuator Dynamics
  - L_1 Adaptive Control for Time-Varying Reference Systems
  - L_1 Adaptive Control for Nonlinear Strict Feedback Systems in the presence of Unmodeled Dynamics
  - L_1 Adaptive Control for Systems with Hysteresis
  - L_1 Adaptive Control for a Class of Systems with Unknown Nonaffine-in-Control Nonlinearities
  - L_1 Adaptive Control for MIMO Systems in the Presence of Unmatched Nonlinear Uncertainties
  - L_1 Adaptive Control in the Presence of Input Quantization
  - ...

- **Output-Feedback:**
  - L_1 Adaptive Output-Feedback Control for Systems of Unknown Dimension (SPR ref. system)
  - L_1 Adaptive Output-Feedback Control for Non-Strictly Positive Real Reference Systems

Guaranteed bounds using L_1-norms and bounded uncertainties/ nonlinearities
Examples including flight test results


**L₁ adaptive output feedback**

![Block diagram of adaptive output feedback system](image)

The control signal is generated according to the following law, assuming zero initialization for $C(s)$:

$$u(s) = C(s)(r(s) - \hat{\sigma}(s)).$$  \hspace{1cm} (4.23)

- LTI controller for all $\Gamma$. 

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EECE 574 - Other Approaches to Adaptive Control  
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What is Adaptive Control?

According to G. Zames:

- A non-adaptive controller is based solely on a-priori information.
- An adaptive controller is based on a posteriori information as well.

The $\mathcal{L}_1$ closed-loop reference model is based solely on a-priori information.

\(^2\) 35th CDC, Kobe, Dec 1996
Conclusions

Current $\mathcal{L}_1$ adaptive controllers do not appear to provide fast adaptation.

- $\mathcal{L}_1$ adaptive controllers are asymptotically equivalent to implementable controller

Open questions

- Fast adaptation vs. robustness?
- High gain feedback vs. adaptation?
- Assumptions and a priori information?
- Performance improvement through linear controller augmentation?
- ...
Let the minimum-phase system be

\[ y(t) = \frac{B(q^{-1})}{A_1(q^{-1})} q^{-k} u(t) + \frac{C(q^{-1})}{A_2(q^{-1})} \Delta d e(t) \]

The minimum-variance controller is

\[ u(t) = -\frac{A_1(q^{-1}) G(q^{-1})}{B(q^{-1}) F(q^{-1}) A_2(q^{-1}) \Delta d} y(t) \]

where

\[ C(q^{-1}) = F(q^{-1}) A_2(q^{-1}) \Delta d + q^{-k} G(q^{-1}) \]

Then

\[ \sigma_{mv}^2 = (1 + f_1^2 + f_2^2 + \cdots + f_{k-1}^2) \sigma_e^2 \]
Assuming the same system is under feedback control with

\[ u(t) = \frac{N(q^{-1})}{D(q^{-1})} \]

The closed-loop system can be described by

\[ y(t) = \frac{CA_1D}{\Delta^d[A_1A_2D - BNA_2q^{-k}]} e(t) \triangleq H(q^{-1})e(t) \]

It takes a few algebraic manipulations to show that

\[ H(q^{-1}) = F(q^{-1}) + q^{-k} \frac{BNA_2\Delta^dF + GA_1D}{\Delta^d[A_1A_2D - BNA_2q^{-k}]} \]