EECE 574 - Adaptive Control

Stochastic Self-Tuning Controllers

Guy Dumont

Department of Electrical and Computer Engineering
University of British Columbia

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Stochastic Self-Tuning Controllers

- Minimum-variance control
- LQG control
- Predictive control
- Indirect self-tuning controllers
- Direct self-tuning controllers

Minimum-Variance Control

- Consider the ARMAX model:

$$A(q)y(t) = B(q)u(t) + C(q)e(t)$$

$$\text{with}\quad \begin{aligned} A(q) &= q^n + a_1 q^{n-1} + \dots a_n \\ B(q) &= b_0 q^{n-d} + b_1 q^{n-d-1} + \dots b_{n-d} \\ C(q) &= q^n + c_1 q^{n-1} + \dots c_n \end{aligned}$$

- Note that $d = \deg A - \deg B$ is the time delay of the process
- As usual, $e(t)$ is assumed to be zero-mean white noise

Minimum-Variance Control

- Omitting the argument q in the various polynomials, the output at time $t+d$ can be written as:

$$y(t+d) = \frac{B}{A}u(t+d) + \frac{C}{A}e(t+d)$$

- The second term in the right-hand side of the above equation consists of noise terms which at time t are future and unknown, and of some past and present terms.
- Because the noise $e(t)$ is white, those future noise terms cannot be predicted, i.e. $E[e(t+j)] = 0 \qquad \text{for } j > 0$.

Minimum-Variance Control

We can express the noise filter C/A by its impulse response model, and separate the unknown future terms from the known past and present ones:

$$\frac{C}{A}e(t+d) = \underbrace{e(t+d)+f_1\,e(t+d-1)+\ldots+f_{d-1}\,e(t+1)}_{\text{future unknown terms}}$$
$$+\underbrace{f_d\,e(t)+f_{d+1}\,e(t-1)+\cdots}_{\text{present and past terms}}$$

Minimum-Variance Control

This can be rewritten as

$$\frac{C}{A}q^{d-1}\,e(t+1) = Fe(t+1) + \frac{qG}{A}e(t)$$

i.e. F and G satisfy the following Diophantine equation with $\deg F = d-1$ and $\deg G = n-1$:

Diophantine Equation

$$q^{d-1}C(q) = A(q)F(q) + G(q)$$

With that the predictor becomes

$$y(t+d) = \frac{B}{A}u(t+d) + (Fq^{-d+1} + \frac{G}{A}q^{-d+1})e(t+d)$$

Minimum-Variance Control

Reconstructing the noise sequence $e(t)$ as $e(t) = [Ay(t) - Bu]/C$ we can write the d-steps ahead predictor

$$y(t+d) = \frac{B}{A}u(t+d) + \frac{qG}{A}e(t) + Fe(t+1)$$

as

$$y(t+d) = \frac{B}{A}u(t+d) + \frac{qG}{A}\frac{Ay(t) - Bu(t)}{C} + Fe(t+1)$$

or

$$y(t+d) = \frac{qG}{C}y(t) + \frac{B(Cq^d - qG)}{AC}u(t) + Fe(t+1)$$

with $q^{d-1}C = AF + G$ this becomes

$$y(t+d) = \frac{qG}{C}y(t) + \frac{qBF}{C}u(t) + F\,e(t+1)$$

Minimum-Variance Control

The predictor can then be written as

$$y(t+d) = \hat{y}(t+d|t) + Fe(t+1)$$

If y_r denotes the setpoint, then the minimum variance controller is obtained by setting

$$\hat{y}(t+d|t) = y_r = \frac{qG}{C}y(t) + \frac{qBF}{C}u(t)$$

thus giving the controller:

MVC Controller

$$u(t) = \frac{C(q)}{B(q)F(q)}\, y_r - \frac{G(q)}{B(q)F(q)}\, y(t)$$

Minimum-Variance Control

Note that when minimum-variance control is used, then

$$\begin{aligned} y(t+d) &= Fe(t+1) \\ &= e(t+d)+f_1e(t+d-1)+\dots \\ &\quad +f_{d-1}e(t+1) \end{aligned}$$

i.e.

- The output is a moving-average process of order $d-1$.
- A characteristic of such a process is that its autocorrelation function vanishes after d lags.
- This is an important feature of minimum-variance control, which will prove very useful when testing the optimality of a stochastic control scheme.

Minimum-Variance Control

- Using this controller gives for the closed-loop system:

$$BCq^{d-1}y(t) = BCy_r + BCFe(t)$$

- Multiplying both sides of the Diophantine equation by $B(q)$, we can interpret the minimum-variance controller as a pole-placement controller.

$$q^{d-1}C(q)B(q) = A(q)F(q)B(q) + G(q)B(q)$$

- The MVC can then be interpreted as a pole placement controller with $B^+ = B$, $A_m = q^{d-1}$ and $A_0 = C$

Minimum-Variance Control

- Like the Dahlin controller, the minimum-variance controller cancels all process zeros, thus no zeros are allowed outside the unit circle, and poorly damped zeros will cause ringing.
- If the B-polynomial is factored as

$$B(q) = B^+(q)B^-(q)$$

with B monic and where all zeros of $B^+(q)$ are within the unit circle, and those of $B^-(q)$ are on or outside the unit circle.

Minimum-Variance Control

The minimum-variance control law is then given by:

$$u(t) = -\frac{G(q)}{B^{+}(q)F(q)}y(t)$$

with $\deg F(q) = d + \deg B^{-} - 1$, $\deg G(q) = n - 1$ and

$$q^{d-1}C(q)B^{-*}(q) = A(q)F(q) + B^{-}(q)G(q)$$

where $B^{-*}(q) = q^{\deg B^{-}}B^{-}(q^{-1})$ projects the unstable zeros inside the unit circle.

Minimum-Variance Control

- Major drawbacks of minimum-variance control
  - Hyperactivity
  - Lack of tuning parameter
- Modifications
  - Moving-average controller
  - Input weighting

$$J = E[y^2 + \rho u^2]$$

$$J = E[y^2 + \rho \Delta u^2]$$

Moving-Average Control

The moving-average controller can be seen as a simplification of the above scheme, where instead of projecting the unstable zeros inside the unit circle by using the reciprocal of B before cancelling them, we simply add an equivalent number of poles at the origin. The Diophantine equation is then

$$q^{h-1}C(q) = A(q)F(q) + B^{-}(q)G(q)$$

or

$$q^{h-1}B^{+}(q)C(q) = A(q)B^{+}(q)F(q) + B(q)G(q)$$

where $h = \deg A - \deg B^{+}$ and $\deg F = h - 1$.

Moving-Average Control

The control law is

$$u(t) = -\frac{G}{B^+F}y(t)$$

The closed-loop system is

$$\begin{aligned} y(t) &= \frac{CB^+F}{q^{h-1}B^+C}e(t) \\ &= q^{-h+1}Fe(t) \\ &= (1+f_1q^{-1}+\ldots+f_{h-1}q^{-h+1})e(t) \end{aligned}$$

The controlled output is then a moving-average of order $h-1$

- If $h = d$ then minimum-variance control is obtained.
- If $h \deg A = n$, then no zeros are cancelled.

Linear Quadratic Control

Theorem (LQG Control)

Consider the process model

$$A(q)y(t) = B(q)u(t) + C(q)e(t)$$

and the loss function

$$J = E\{(y(t) - y_r(t))^2 + \rho u^2(t)\}$$

Assume that there is no common factor to all three polynomials $A(q)$, $B(q)$ and $C(q)$. The optimal control law that minimizes J is given by

$$R(q)u(t) = -S(q)y(t) + T(q)y_r(t)$$

where $T(q) = t_0 C(q)$ with $t_0 = P(1)/B(1)$ and R and S satisfy the Diophantine equation

$$A(q)R(q) + B(q)S(q) = P(q)C(q)$$

where the monic and stable $P(q)$ is obtained from the spectral factorization

$$rP(q)P(q^{-1}) = \rho A(q)A(q^{-1}) + B(q)B(q^{-1})$$

For proof, see Aström and Wittenmark, Computer Controlled Systems

Linear Quadratic Control

- The LQG controller is obtained as solution of the Diophantine equation

$$P(q)C(q) = A(q)R(q) + B(q)S(q)$$

- The closed-loop characteristic polynomial is $P(q)C(q)$ where the stable polynomial $P(q)$ is obtained from the following spectral factorization:

$$rP(q)P(q^{-1}) = \rho A(q)A(q^{-1}) + B(q)B(q^{-1})$$

Linear Quadratic Control

- If A and B have a stable common factor A_2, i.e. $A = A_1A_2$ and $B = B_1A_2$ then P can be written as $P = P_1A_2$ where P_1 satisfies

$$rP_1(q)P_1(q^{-1}) = \rho A_1(q)A_1(q^{-1}) + B_1(q)B_1(q^{-1})$$

  and

$$P = P_1A_2$$

- The Diophantine equation to be solved then becomes

$$P_1(q)C(q) = A_1(q)R(q) + B_1(q)S(q)$$

- A_2 contains the uncontrollable modes

Linear Quadratic Control

- More complicated when some uncontrollable modes are unstable, i.e. when $A_2(q) = A_2^+(q)A_2^-(q)$
- With a controller $u(t) = -S(q)/R(q)y(t)$ the closed-loop input and output are described by

$$y(t) = \frac{R(q)C(q)}{A(q)R(q)+B(q)S(q)}e(t)$$

$$u(t) = \frac{S(q)C(q)}{A(q)R(q)+B(q)S(q)}e(t)$$

- The output y will be bounded if $R(q)$ contains $A_2^-(q)$
- The input u will be bounded if $s(q)$ contains $A_2^-(q)$
- Because R and S are co-prime, only y will be bounded by making sure that $R(q)$ contains $A_2^-(q)$ as a factor. This is the internal model principle.

Indirect MV-STR

Straightforward design

- Use e.g. AML to obtain estimates $\hat{A}$, $\hat{B}$ and $\hat{C}$
- Compute zeros of $\hat{C}$ and project unstable ones inside the unit circle
- Factor $\hat{B}$ as $\hat{B} = \hat{B}^{+}\hat{B}^{-}$
- Eliminate common factors between $\hat{A}$ and $\hat{B}^{-}$
- Solve the Diophantine equation

$$q^{d-1}\hat{C}(q)\hat{B}^{-*}(q) = \hat{A}(q)F(q) + \hat{B}^{-}(q)G(q)$$

  where $\deg F(q) = d + \deg \hat{B}^{-} - 1$, $\deg G(q) = n - 1$ and $\hat{B}^{-*}(q) = q^{\deg \hat{B}^{-}}\hat{B}^{-}(q^{-1})$

- Apply the minimum variance controller

$$u(t) = -\frac{G(q)}{\hat{B}^{+}(q)F(q)}y(t)$$

Indirect MV-STR Example

- Consider the system

$$y(t+1)+ay(t)=bu(t)+e(t+1)+ce(t)$$

  with $a=-0.9$, $b=3$, and $c=-0.3$.

- The minimum variance controller is

$$u(t)=\frac{a-c}{b}y(t)=-s_0y(t)=-0.2y(t)$$

- Initial estimates $\hat{a}(0)=\hat{c}(0)=0$, $\hat{b}(0)=1$

Indirect MV-STR Example

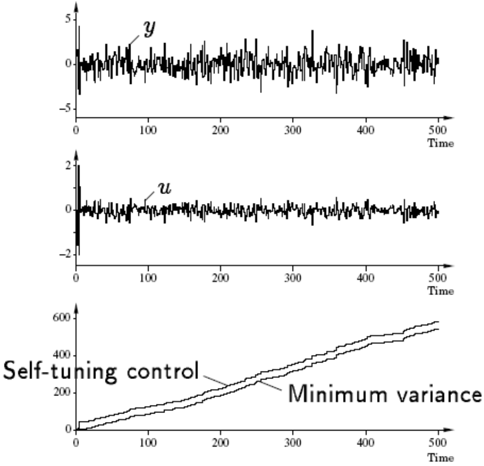


Figure: Indirect MV-STR

Indirect LQG-STR

- Estimate $\hat{A}$, $\hat{B}$ and $\hat{C}$ using e.g. AML
- Compute zeros of $\hat{C}$ and project unstable ones inside the unit circle
- Eliminate common factors between $\hat{A}$, $\hat{B}$ and $\hat{C}$
- Find the common factor A_2 between $\hat{A}$ and $\hat{B}$
- Given ρ, solve the appropriate spectral factorization problem
- Solve the appropriate Diophantine equation
- Implement control law with computed R, S, and T

The solution of the spectral factorization problem is the major computation in the LQG STR. It is crucial to use a method that garantees a stable P.

Direct MV-STR Example

- Consider again the system

$$y(t+1)+ay(t)=bu(t)+e(t+1)+ce(t)$$

with $a=-0.9$, $b=3$, and $c=-0.3$.

- The minimum variance controller is

$$u(t)=\frac{a-c}{b}y(t)=-s_0y(t)=-0.2y(t)$$

- Consider a direct STR based on the model

$$y(t+1)=r_0u(t)+s_0y(t)$$

for which the control law is

$$u(t)=-\frac{\hat{s}_0}{\hat{r}_0}y(t)$$

with $\hat{r}_0=1$ and $\hat{s}_0$ is estimated with a simple RLS.

Direct MV-STR Example

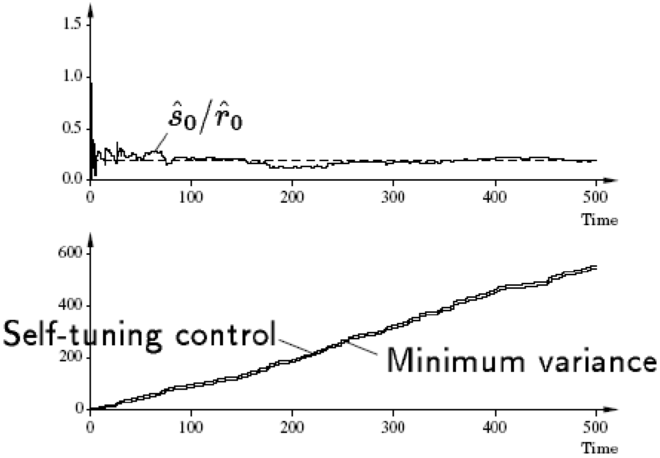


Figure: Indirect MV-STR

Direct Stochastic Self-Tuning Controllers

- The previous indirect scheme involves many calculations at each step and requires the estimation of the C-polynomial
- Direct schemes are much more economical from a computational viewpoint
- Direct STR involves reparameterizing the problem in terms of the controller parameters
- Key to this reparameterization is the predictor form

Predictor Models

The predictor for $y(t+d)$ derived for minimum-variance control can be written in terms of polynomials in the backward shift operator q^{-1} as

$$y(t+d) = \frac{1}{C}\left(Gy(t) + BFu(t)\right) + Fe(t+d)$$

or

$$y(t+d) = S(q^{-1})y_f(t) + R(q^{-1})u_f(t) + F(q^{-1})e(t+d)$$

with

$$y_f(t) = \frac{y(t)}{C(q^{-1})} \qquad y_f(t) = \frac{y(t)}{C(q^{-1})}$$

In this predictor model, the controller polynomials R and S appear directly.

Direct MV-STR

- Given n and d, then $\deg R = n+d-1$ and $\deg S = n-1$. Let $1/C^*(q^{-1})$ be a stable filter
- Get estimates $\hat{R}$ and $\hat{S}$ from

$$y(t+d) = S(q^{-1})y_f(t) + R(q^{-1})u_f(t) + \varepsilon(t+d)$$

where

$$y_f(t) = \frac{y(t)}{C(q^{-1})} \qquad y_f(t) = \frac{y(t)}{C(q^{-1})}$$

using RLS with

$$\varphi^T(t) = \frac{1}{C}[u(t) \ldots u(t-n-d+1) y(t) \ldots y(t-n+1)]$$

$$\theta^T = [r_0 \ldots r_{n+d-1} s_0 \ldots s_{n-1}]$$

- Apply the control law

$$\hat{R}(q^{-1})u(t) = -\hat{S}(q^{-1})y(t)$$

Properties of the Direct MV-STR

Assume direct MV-STR with $C^*(q^{-1}) = 1$

Property 1: If regression vectors are bounded, then the closed-loop system is such that

$$\overline{y(t+\tau)y(t)} = 0 \qquad \tau = d, \dots d+n-1$$

$$\overline{y(t+\tau)u(t)} = 0 \qquad \tau = d, \dots d+n+d-1$$

Property 2: If the direct MV-STR is applied to

$$A(q)y(t) = B(q)u(t) + C(q)e(t)$$

then

$$\overline{y(t+\tau)y(t)} = 0 \qquad \tau = d, \dots$$

Thus, if the direct MV-STR converges, and if there are sufficiently many parameters in the controller, then it converges to the minimum-variance controller.

Some Remarks

The predictor can be written as

$$y(t+d) = \varphi^T(t)\hat{\theta}(t) + \varepsilon(t+d)$$

and the MVC controller as

$$\varphi^T(t)\hat{\theta}(t) = 0$$

Remark 1: Note that $k\varphi^T(t)\hat{\theta}(t) = 0$ also gives MV control. Parameters have one degree of freedom and may thus wander in unison (they lie on a linear manifold). Unique estimation may be obtained by fixing e.g. r_0. Ideally, it should be equal to b_0

Remark 2: For convergence, it is necessary that $\{u(t)\}$ be persistently exciting of order $\geq 2n+d$. How can we guarantee that in adaptive control? How can we even guarantee stability? These are non-trivial questions.

Some Remarks

Remark 3: It is easy to add feedforward and command signals

$$y(t+d) = S(q^{-1})y_f(t) + R(q^{-1})u_f(t) + S_{ff}(q^{-1})v_f(t) + \varepsilon(t+d)$$

where v_f is the filtered measured disturbance.

The controller is then

$$\hat{R}(q^{-1})u(t) = -\hat{S}(q^{-1})y(t) - \hat{S}_{ff}(q^{-1})v(t)$$

Can also add setpoint tracking.

Remark 4: By replacing d by $h > d$, we obtain a self-tuning moving-average controller. By sufficiently large h, no zeros are cancelled, and non-minimum phase systems can be controlled.

Direct MA-STR

- Consider an integrator with time delay τ sampled with sampling period h.

$$
\begin{aligned}
A(q) &= q(q-1) \\
B(q) &= (h-\tau)(q+\frac{\tau}{h-\tau}) \\
C(q) &= q(q+c)
\end{aligned}
$$

- Minimum phase if $\tau < h/2$.
- With $h = 1$, at time $t = 100$, the delay is changed from 0.4 to 0.6.
- Simulation with $d = 1$ (MV-STR) and $d = 2$ (MA-STR).

Direct MA-STR Example

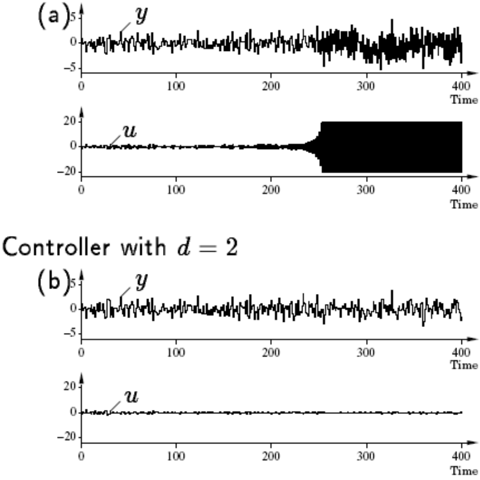


Figure: Direct MV-STR and MA-STR

Generalized MV Self-Tuners

- Extended Minimum-Variance Controller
- Generalized Minimum-Variance Controller

Extended Minimum-Variance Control

Clarke and Gawthrop, "A Self-Tuning Controller", IEE Proc. **122:**929-934

Known Parameters

Define the auxiliary variable

$$\phi(t) \triangleq Py(t) - Rq^{-d}y_r(t) + Qq^{-d}u(t)$$

where P Q and R are polynomials.

Multiplying by A and assuming that $y(t)$ is governed by an ARMAX process gives

$$A\phi(t) = P[Bq^{-d}u(t) + Ce(t)] - q^{-d}ARy_r(t) + Qq^{-d}Au(t)$$

$$A\phi(t) = q^{-d}[PB + QA]u(t) - q^{-d}ARy_r(t) + PCe(t)$$

or

$$\phi(t) = \frac{q^{-d}[PB + QA]}{A}u(t) - q^{-d}Ry_r(t) + \frac{PC}{A}e(t)$$

Extended Minimum-Variance Control

Consider the Diophantine equation

$$PC = AF + q^{-d}G$$

where $\deg F = d - 1$ and $\deg G = n + n_p - 1$ With this, we can write a predictor for $\phi(t+d)$:

$$\phi(t+d) = \frac{BF+QC}{C}u(t) + \frac{G}{C}y(t) - Ry_r(t) + Fe(t+d)$$

or

$$\phi(t+d) = \hat{\phi}(t+d|t) + \varepsilon(t+d)$$

with

$$\hat{\phi}(t+d|t) = \frac{1}{C}[(BF+QC)u(t) + Gy(t) - CRy_r(t)]$$

Extended Minimum-Variance Control

The error term $\varepsilon(t+d) = Fe(t+d)$ is uncorrelated with $\hat{\phi}(t+d|t)$. We can thus proceed as in minimum variance control. The cost function

$$J = E[\phi^2(t+d)]$$

is minimized by setting $\hat{\phi}(t+d|t) = 0$, i.e.

$$(BF+QC)u(t) + Gy(t) - CRy_r(t) = 0$$

or

EMVC

$$u(t) = -\frac{G}{BF+QC}y(t) + \frac{CR}{BF+QC}y_r(t)$$

then

$$\phi(t+d) = Fe(t+d)$$

The above control law minimizes $E[\phi^2(t+d)]$, i.e. is the MV controller for ϕ.

Extended Minimum-Variance Control

The closed-loop system is

$$y(t) = \frac{B}{A} q^{-d} \left[\frac{CRy_r(t) - Gy(t)}{BF + QC} \right] + \frac{C}{A} e(t)$$

or

$$C[BP + QA]y(t) = BCRq^{-d} y_r + c[BF + QC]e(t)$$

The closed-loop characteristic polynomial is

$$C[BP + QA]$$

EMVC can handle non-minimum phase systems.

Extended Minimum-Variance Control

The EMVC control law that minimizes

$$E[\phi^2(t+d)] = E[(Py(t+d) - Ry_r + Qu(t))^2]$$

also minimizes

$$E[(Py(t+d) - Ry_r)^2 + \mu(Qu(t))^2] \qquad \mu = \frac{b_0}{q_0}$$

i.e. this can be interpreted as LQG!
Proof: Consider

$$\begin{aligned} I &= E[(Py(t+d) - Ry_r)^2 + \mu(Qu(t))^2] \\ &= ((P\hat{y}(t+d|t) - Ry_r)^2 + \mu(Qu(t))^2 + \sigma_\varepsilon^2 \end{aligned}$$

Then

$$\frac{\partial I}{\partial u(t)} = 2(P\hat{y}(t+d|t) - Ry_r)b_0 + 2\mu q_0 Qu(t) = 0$$

Implies

$$P\hat{y}(t+d|t) + \mu\frac{q_0}{b_0}Qu(t) - Ry_r = 0$$

which is the same control as before if $\mu = b_0/q_0$.

Extended Minimum-Variance Control

Note that zero steady-state error requires

$$\left.\frac{BR}{PB+QA}\right|_{z=1} = 1$$

This is satisfied by choosing $R = P(1)$ and $Q(1) = 0$

- With $P = R = 1$ and $Q = \lambda$

$$J = E[(y(t+d) - y_r)^2 + \lambda' u^2(t)]$$

- With $P = R = 1$ and $Q = \lambda(1-q^{-1})$

$$J = E[(y(t+d) - y_r)^2 + \lambda' \Delta u^2(t)]$$

  the controller then contains an integrator.

- With $P = R = 1$ and $Q = 0$

$$J = E[(y(t+d) - y_r)^2]$$

  MVC with setpoint tracking.

Direct EMV-STC

A direct EMV self-tuner can be implemented by estimating directly the parameters of the predictor form:

$$C\hat{\phi}(t+d|t) = (BF+QC)u(t) + Gy(t) - CRy_r(t)$$

Note that this seemingly requires direct estimation of the C polynomial. However, the predictor

$$C\phi(t+d) = (BF+QC)u(t) + Gy(t) - CRy_r(t) + FCe(t+d)$$

can be rewritten as

$$[C+(1-C)]\phi(t+d) = (BF+QC)u(t) + Gy(t) - CRy_r(t) + FCe(t+d) + (1-C)\phi(t+d)$$

$$\phi(t+d) = (BF+QC)u(t) + Gy(t) - CRy_r(t) + Fe(t+d) + (1-C)[\phi(t+d) - Fe(t+d)]$$

Under the previously derived control law, $\phi(t+d) = Fe(t+d)$ and the last term of the above equation vanishes.

Note that it only says that convergence to the correct control law is possible even without knowing C, but it does not establish that convergence. This will be discussed later.

Example

$$y(t) = 2y(t-1) + u(t-2) + 2u(t-3) + e(t) \text{ where } \sigma_e^2 = 0.5$$

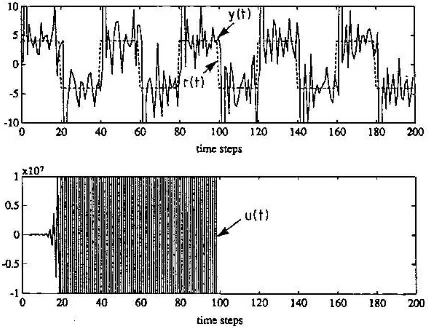


Figure: MVC

Example

$$y(t) = 2y(t-1) + u(t-2) + 2u(t-3) + e(t) \text{ where } \sigma_e^2 = 0.5$$

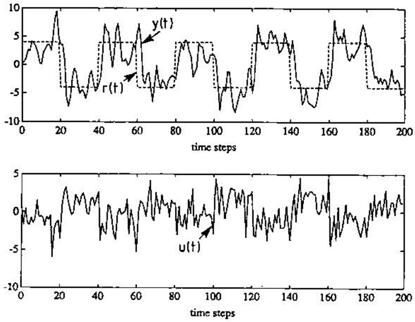


Figure: EMVC, $P = Q = 1$, $BP + AQ = 2$

Example

$$y(t) = 2y(t-1) + u(t-2) + 2u(t-3) + e(t) \text{ where } \sigma_e^2 = 0.5$$

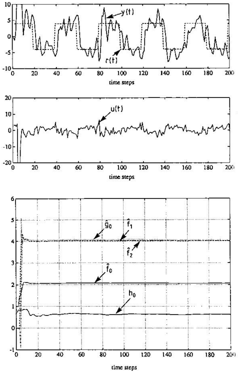
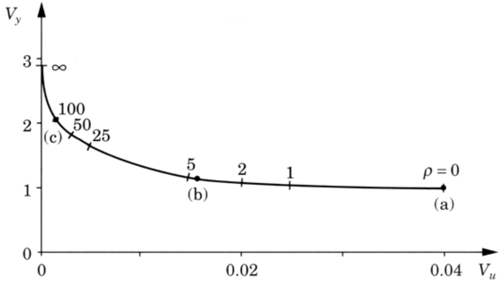


Figure: Adaptive EMVC, $P = Q = 1$, $BP + AQ = 2$

EMVC Example

$y(t+1)+ay(t) = bu(t)+e(t+1)+ce(t)$ with $a = -0.9$, $b = 3$, and $c = -0.3$.
The output variance as a function of the input variance is as shown below



EMVC Example